Optimal design for accelerated degradation tests with stochastic model uncertainty

LE LIU, XIAO-YANG LI, TONG-MIN JIANG & JIAN-GUO ZHANG

1. School of Reliability and Systems Engineering, Beihang University, Beijing, China
2. Science and Technology on Reliability and Environmental Engineering Laboratory, Beijing, China

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- Literature review.
- Model uncertainty in ADT planning.
- Bayesian model averaging method.
- Optimization problem.
- Illustrative example & sensitivity analysis.
- Conclusion & future work.
Literature Review

- How to design ADT plan? – Target & decision variables.
- What is the model ADT data? Degradation path or stochastic process.
- How to quantify model uncertainty in ADT planning, i.e. Wiener process, Gamma process and inverse Gaussian process?
- Are there any relevant researches on this topic?

1. Total cost
2. Estimation precision of the lifetime
3. Etc.

1. Sample size allocation
2. Inspection times
3. Stress levels
4. Etc.
Model uncertainty

- **Stochastic process models for degradation modeling**
  - Wiener process (M1)
  - Gamma process (M2)
  - Inverse Gaussian process (M3)

**Basic assumptions:**

- \( \forall t_2 > t_1 \geq s_2 > s_1, X(t) \) has independent increments, that is, \( X(t_2) - X(t_1) \) and \( X(s_2) - X(s_1) \) are independent.
- \( \forall t \geq s > 0, X(t) - X(s) \) follows certain distributions, whose mean and variance are proportional to \( \Lambda(t) - \Lambda(s) \).

where \( \Lambda(t) \) is the time-scale transformation.
Model uncertainty

- **Unified stochastic process model** \( USP(a, b) \)

\[
E[X] = \mu \Lambda(t) \quad \text{Var}[X] = \sigma^2 \Lambda(t)
\]

<table>
<thead>
<tr>
<th>Model</th>
<th>( a )</th>
<th>( b &gt; 0 )</th>
<th>PDF</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_1 )</td>
<td>( \mu \Lambda(t) )</td>
<td>( \sqrt{\sigma^2 \Lambda(t)} )</td>
<td>( f_N(x</td>
</tr>
<tr>
<td>( M_2 )</td>
<td>( \mu^2 \Lambda(t)/\sigma^2 )</td>
<td>( \sigma^2/\mu )</td>
<td>( f_{Ga}(x</td>
</tr>
<tr>
<td>( M_3 )</td>
<td>( \mu \Lambda(t) )</td>
<td>( \mu^3 \Lambda^2(t)/\sigma^2 )</td>
<td>( f_{IG}(x</td>
</tr>
</tbody>
</table>
Model uncertainty

- **Acceleration model**

\[ \mu_i = \exp(\alpha_0 + \alpha_1 s_i) \]

where \( s_i \in [0,1] \) is the normalized accelerated stress level by

\[ s_i = \begin{cases} 
\frac{1/s_0' - 1/s_i'}{1/s_0' - 1/s_H'} & \text{Arrhenius relation} \\
\frac{\ln s_i' - \ln s_0'}{\ln s_H' - \ln s_0'} & \text{Power law relation} \\
\frac{s_i' - s_0'}{s_H' - s_0'} & \text{Exponential relation}
\end{cases} \]

where \( s_0' \) and \( s_H' \) are the normal and highest stress levels, \( i = 1, 2, \ldots, K \).

The degradation rate at the use condition is

\[ \mu_0 = \exp(\alpha_0). \]
Model uncertainty

- The $p$-th quantile lifetime at the use condition

$$t_p = \Lambda^{-1} \left[ \frac{\beta}{4} \left( \alpha z_p + \sqrt{4 + \alpha^2 z_p^2} \right)^2 \right]$$

where $z_p$ is the $p$-quantile of the standard normal distribution, and $\Lambda^{-1}(\cdot)$ is the inverse function of $\Lambda(\cdot)$.

Based on the asymptotically normal distribution, the asymptotic variance of $t_p$ can be given as

$$Avar(t_p) = (\nabla t_p)' I^{-1}(\theta) \nabla t_p$$

where $I(\theta)$ is the expected Fisher information matrix.
Bayesian model averaging (BMA)

- **Uncertainty quantification**

Considered that $\Delta$ is the quantity of interest, i.e. $t_p$ at the use condition, its posterior distribution given data $D$ is

$$f(\Delta|D) = \sum_{c=1}^{C} f(\Delta|M_c, D) Pr\{M_c|D\}$$

where $f(\Delta|M_c, D)$ is the posterior density of $\Delta$ assuming that $M_c$ is the true model, $Pr\{M_c|D\}$ is the posterior probability of model $M_c$.

$$Pr\{M_c|D\} \propto f(D|M_c) Pr\{M_c\}$$

where $Pr(M_c)$ is the prior probability of model $M_c$, $f(D|M_c)$ is the integrated likelihood of model $M_c$

$$f(D|M_c) = \int f(D|\theta_c, M_c) f(\theta_c|M_c)$$

where $f(D|\theta_c, M_c)$ is the likelihood function under model $M_c$. 

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Bayesian model averaging (BMA)

- **Uncertainty quantification**

  For \( \Delta \), its posterior mean is
  
  \[
  E(\Delta|D) = \sum_{c=1}^{C} \tilde{\Delta}_c \Pr\{M_c|D\}
  \]

  and its variance is
  
  \[
  Var(\Delta|D) = \sum_{c=1}^{C} \left( Var(\Delta|M_c, D) + \tilde{\Delta}_c \right) \Pr\{M_c|D\} - E(\Delta|D)^2
  \]

  where \( \tilde{\Delta}_c = E(\Delta|M_c, D) \).
Optimization problem

Single model

\[
\text{Minimize} \quad \text{Var}(\hat{r}_p)
\]

subject to

\[
0 \leq s_i \leq 1, \quad s_{i-1} \leq s_i, \quad i \in [1, K]
\]

\[
\sum_{i=1}^{K} n_i = N, \quad n_i \leq n_{i-1}, \quad i \in [2, K]
\]

\[
\sum_{i=1}^{K} m_i = M, \quad m_i \leq m_{i-1}
\]

\(n_i\) and \(m_i\) \in \mathbb{N}^+

Multiple models

\[
\text{Minimize} \quad \text{Var}[\hat{r}_p|D]
\]

subject to

\[
0 \leq s_i \leq 1, \quad s_{i-1} \leq s_i, \quad i \in [1, K]
\]

\[
\sum_{i=1}^{K} n_i = N, \quad n_i \leq n_{i-1}, \quad i \in [2, K]
\]

\[
\sum_{i=1}^{K} m_i = M, \quad m_i \leq m_{i-1}
\]

\(n_i\) and \(m_i\) \in \mathbb{N}^+
Illustrative example

Stress relaxation CSADT data

<table>
<thead>
<tr>
<th>Contents</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accelerated stresses</td>
<td>65, 85, 100°C</td>
</tr>
<tr>
<td>Normal condition</td>
<td>40°C</td>
</tr>
<tr>
<td>Failure threshold</td>
<td>30%</td>
</tr>
</tbody>
</table>
Illustrative example

- Model selection with $L_{\text{max}}$ value

<table>
<thead>
<tr>
<th>Models</th>
<th>$\alpha_0$</th>
<th>$\alpha_1$</th>
<th>$\sigma$</th>
<th>$\gamma$</th>
<th>$L_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>-3.4857</td>
<td>2.8951</td>
<td>0.4222</td>
<td>0.2997</td>
<td>-74.6</td>
</tr>
<tr>
<td>$M_2$</td>
<td>-2.9438</td>
<td>1.0417</td>
<td>0.1922</td>
<td>0.4381</td>
<td>55.1</td>
</tr>
<tr>
<td>$M_3$</td>
<td>-3.6858</td>
<td>1.2253</td>
<td>0.2606</td>
<td>0.5181</td>
<td>54.8</td>
</tr>
</tbody>
</table>

- this process maybe ignored in real applications with the assumption of the degradation model, or
- this selection maybe inappropriate for designing ADT plans without the consideration of model uncertainty.
Model comparison for the 2-level CSADT plan

Assumptions:

- $N = 10$ samples
- $M = 100$ inspection
- Time interval 24 hours
- $p = 0.1$

- the optimal second stress level is the same for three candidate model ($s_2 = 1$, i.e. $100\, ^\circ C$).
- the sample allocations, inspection times and the first stress level are different with different choice of the degradation model.
- $M_2$ is treated as the most suitable model from Table 2 but without the highest prediction precision.

<table>
<thead>
<tr>
<th>Models</th>
<th>Samples</th>
<th>Inspection</th>
<th>Stress</th>
<th>std($\hat{t}_p$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>8</td>
<td>2</td>
<td>96</td>
<td>4</td>
</tr>
<tr>
<td>$M_2$</td>
<td>7</td>
<td>3</td>
<td>84</td>
<td>16</td>
</tr>
<tr>
<td>$M_3$</td>
<td>5</td>
<td>5</td>
<td>67</td>
<td>33</td>
</tr>
<tr>
<td>$BMA$</td>
<td>7</td>
<td>3</td>
<td>82</td>
<td>18</td>
</tr>
</tbody>
</table>

Pr($M_1 | D$) = 0
Pr($M_2 | D$) = 0.5628
Pr($M_3 | D$) = 0.4372
Sensitivity analysis

- Variation of parameters

Setting true values as $a_0 = 2$, $a_1 = 1.5$, $\sigma = 0.5$, $\gamma = 0.4$ and the variations into three levels, i.e. +10%, 0, -10%. $Pr(M_c | D) = 1/3$.

![Table 4: Sensitivity analysis of the variation of parameters on the optimal 2-level CSADT plan.](image)
Sensitivity analysis

- Variation of model posterior probabilities

Setting true values as $\alpha_0 = 2$, $\alpha_1 = 1.5$, $\sigma = 0.5$, $\gamma = 0.4$ and the variations into three levels, i.e. +10%, 0, -10%.

| $Pr(M_1|D)$ | $Pr(M_2|D)$ | $Pr(M_3|D)$ | $n_1$ | $n_2$ | $m_1$ | $m_2$ | $s_1$ | $s_2$ | $std(t_p)$ |
|-------------|-------------|-------------|-------|-------|-------|-------|-------|-------|------------|
| 1/6         | 1/3         | 1/2         | 6     | 4     | 75    | 25    | 0.49  | 1     | 1.86e5     |
| 1/6         | 1/2         | 1/3         | 7     | 3     | 79    | 21    | 0.51  | 1     | 1.90e5     |
| 1/3         | 1/6         | 1/2         | 7     | 3     | 83    | 17    | 0.45  | 1     | 1.97e5     |
| 1/3         | 1/2         | 1/6         | 7     | 3     | 83    | 17    | 0.45  | 1     | 2.03e5     |
| 1/2         | 1/6         | 1/3         | 8     | 2     | 88    | 12    | 0.39  | 1     | 2.07e5     |
| 1/2         | 1/3         | 1/6         | 8     | 2     | 88    | 12    | 0.39  | 1     | 2.10e5     |
Summary

- The BMA method is introduced to analyze the problem of stochastic model uncertainty on designing the optimal ADT plan.
- The simulated stress relaxation CSADT data shows that each stochastic degradation model can produce different optimal plan and prediction precision.
- The sensitivity study shows that the optimal plan is less sensitive to the variation of parameters but to model posterior probabilities, while that for the prediction precision is reverse.
Future Work

- The study of the random effects due to the variation of the tested samples
- Maybe the measurement error
- Competitive study with reference model
References


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