Statistical analysis of Accelerated life testing under Weibull distribution based on fuzzy theory

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SUMMARY & CONCLUSIONS

Sometimes the accelerated life testing (ALT) of some products still take time and it is hard to implement real-time monitoring on samples to check whether the samples fail or not. Under this condition, periodic inspection is always conducted to monitor the sample performance, obtaining the failure time data in ALT. But periodic inspection may cause problems in some cases.

When the failure is detected during the inspection, its timing point is recorded by estimation. So the failure time is not accurate since it may happen at any time in the adjacent inspection interval and cannot be detected immediately after the failure happen. The extreme case is that the sample fails immediately when the inspection is accomplished. Thus, there exists big uncertainty in failure time data if the interval is quite long which will affect the accuracy of evaluation results.

However, there are some methods which may help to detect the failure happen indirectly. For example, the fluctuation or excursion of the output digital signal of samples may indicate the failure to some degree. Even so, the failure time cannot be recognized directly because the judging criteria of failure of different inspectors may slightly vary. This leads to the estimation of the exact failure time is not precise.

Therefore, the failure time data of ALT is not precise and should be fuzzy data. The life estimation of ALT should be a fuzzy interval value rather than just a single number. Hence, in this paper, a statistical method for constant-stress ALT with Type II censored samples based on fuzzy theory is proposed which assumes that the life time of the product follows Weibull distribution.

1 INTRODUCTION

Accelerated life testing has been internationally researched as it saves time and money than traditional life testing. Many statistical methods of ALT have been proposed by researchers around the world. Maximum likelihood estimation (MLE) is one of commonly used method to analyze the failure time data. Fan [1] discusses a parameter estimation method of the generalized gamma lifetime distribution under the constant ALT based on MLE. Chandra [2] provides an optimum plan for step-stress ALT while maximum likelihood function is derived.

As discussed above, the failure data of ALT is not precise and kind of epistemic uncertainty. Fuzzy theory is one of the most important methods dealing with uncertainty. Huang [3] proposes a statistical model on competitive failure process when the degradation data is fuzzy. Jamkhaneh [4] analyzes the system reliability with fuzzy Weibull distribution. Wu [5] provides a Bayesian approach to estimate system fuzzy reliability. Viertl [6] particularly describes some statistical models with fuzzy data.

Therefore, fuzzy theory is chosen together with MLE method to establish a statistical model of constant ALT with Weibull distribution.

2 ACCELERATED LIFE TESTING ANF FUZZY THEORY

2.1 The statistical model of accelerated life testing

The statistical inference of ALT bases on two assumptions. Firstly, the lifetime distribution of the product follows Weibull distribution under the normal stress level and accelerated stress levels. It can be expressed as

$$F(t) = 1 - \exp\left\{-\left(\frac{t}{\eta}\right)^m\right\} \tag{1}$$

And its probability density function is

$$f(t) = \frac{m}{\eta}\left(\frac{t}{\eta}\right)^{m-1} \exp\left[-\left(\frac{t}{\eta}\right)^m\right] \tag{2}$$

In Eq.(1) and Eq.(2),η is scale parameter; m is shape parameter. Secondly, the relationship between the scale parameter η, also

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known as the characteristic life, and the stress is below:

$$\ln \eta_i = a + b\varphi(S_i) \tag{3}$$

In Eq.(3) ,$\varphi(S_i)$ is a function defined by the stress inflicted on the samples.

In CSALT, assuming there are n samples working at constant but different stress levels. Presuming an ALT has k accelerated stress levels, denoted as $S_1<S_2<\ldots<S_k$, normal stress level is S_0. At each accelerated stress level S_i, n_i specimens are run to failure until r_i specimens failed. The failure time data are called Type II censored data. Mark failure times of jth sample under its accelerated stress as t_{ij}, i=1,2,…,k, j=1,2, …, n_i.

Statistical model of constant ALT based on MLE with Type II censored data has been deeply researched. The joint density function is

$$L(a,b,m) = \prod_{i=1}^{k}\prod_{j=1}^{r_i} \frac{n_i!}{(n_i-r_i)!}\frac{m}{\eta_i^m} t_{ij}^{m-1}\exp\left[-\left(\frac{t_{ij}}{\eta_i}\right)^m\right] \cdot \left\{\exp\left[-\left(\frac{t_{r_i}}{\eta_i}\right)\right]\right\}^{n_i-r_i} \tag{4}$$

Eq.(5) is Eq.(4) in logs.

$$\begin{aligned}&\ln L(a,b,m) = \\ &\sum_{i=1}^{k}\left[\ln\frac{n_i!}{(n_i-r_i)!} + r_i\ln m - mr_i\ln\eta_i + (m-1)\sum_{j=1}^{r_i}\ln t_{ij}\right] \\ &-\sum_{i=1}^{k}\left[\sum_{j=1}^{r_i}\left(\frac{t_{ij}}{\eta_i}\right)^m + (n_i-r_i)\left(\frac{t_{r_i}}{\eta_i}\right)^m\right]\end{aligned} \tag{5}$$

After further derivation, Eq.(6)-(8) can be obtained. Combining Eq.(3) and using Newton Raphson iteration method, the parameter a,b and m can be calculated.

$$\sum_{i=1}^{k}\left\{\left[\sum_{j=1}^{r_i}\left(\frac{t_{ij}}{\eta_i}\right)^m + (n_i-r_i)\left(\frac{t_{r_i}}{\eta_i}\right)^m\right] - r_i\right\} = 0 \tag{6}$$

$$\sum_{i=1}^{k}\varphi_i\left\{\left[\sum_{j=1}^{r_i}\left(\frac{t_{ij}}{\eta_i}\right)^m + (n_i-r_i)\left(\frac{t_{r_i}}{\eta_i}\right)^m\right] - r_i\right\} = 0 \tag{7}$$

$$\frac{\sum_{i=1}^{k} r_i}{m} + \sum_{i=1}^{k}\left[\sum_{j=1}^{r_i}\ln t_{ij} - \sum_{j=1}^{r_i}\left(\frac{t_{ij}}{\eta_i}\right)^m \ln t_{ij} + (n_i-r_i)\left(\frac{t_{r_i}}{\eta_i}\right)^m \ln t_{r_i}\right] = 0 \tag{8}$$

2.2 Fuzzy theory

A fuzzy number $\tilde{x}$ is described by a characterizing function, also called membership function $\mu(.)$ which denotes the degree of membership of element $\tilde{x}$ of the universe X.

$$\begin{aligned}&\tilde{x}: X \to 0:1 \\ &\mu_{\tilde{x}}(X) \in [0,1]\end{aligned} \tag{9}$$

An α-cut of $\tilde{x}$, written as $\tilde{x}_\alpha$, is defined as

$$\begin{aligned}&\tilde{x}_\alpha = \left\{x \middle| \mu_{\tilde{x}}(X) \ge \alpha\right\} \\ &0 \le \alpha \le 1\end{aligned} \tag{10}$$

As the value of α is settled, a fuzzy number $\tilde{x}$ can be turned into a finite closed interval as [7]

$$\tilde{x}_\alpha \in \left[\tilde{x}_\alpha^L, \tilde{x}_\alpha^U\right] \tag{11}$$

Considering that the recorded failure time is not precise and the failure happens between the adjacent inspection intervals, such a membership function is chosen in this paper.

$$\mu_{\tilde{t}_{ij}}(t_{ij}) = \left\{\begin{array}{ll} 1-\dfrac{m_{t_{ij}}-t_{ij}}{g_l} & l \le t_{ij} < m_{t_{ij}} \\ 1 & t_{ij} = m_{t_{ij}} \\ 1-\dfrac{t_{ij}-m_{t_{ij}}}{g_u} & m_{t_{ij}} < t_{ij} \le r \\ 0 & else \end{array}\right\} \tag{12}$$

while

$$g_l = m_{t_{ij}} - l, \quad g_u = r - m_{t_{ij}}$$

In Eq.(12), t_{ij} represents the exact failure time of samples. $m_{t_{ij}}$ represents the recorded failure time of samples by researchers. l and r represent the boundary value of the ith inspection interval when the t_{ij} failure is detected. g_l or g_u represents the gap between the recorded time and the edge of the inspection interval. As the Eq.(12) shows, if $\mathrm{m}_{\mathrm{t_{ij}}}$ is precise, the degree of membership of the exact failure time t_{ij} is 1. If the recorded time $\mathrm{m}_{\mathrm{t_{ij}}}$ far from the exact failure time t_{ij}, the degree of membership approaches to zero.

3 THE OPTIMIZAITON MODEL OF ALT

As discussed in the first part, the recorded failure time data is fuzzy and has a membership function like Eq.(11). Therefore, all the t_{ij} in the Eq.(6)-(8) should be changed into $\tilde{t}_{ij}$.

3.1 Parameter optimization of ALT based on PSO

Particle Swarm Optimization(PSO) is a kind of optimization algorithm. By using times of iterations PSO finds the optimal value when given an initial value. In this paper, the failure time data becomes a finite closed fuzzy interval value. When t_{ij} changes in its interval, the value of the parameter a,b and m will be change as a result. The extension theorem of fuzzy theory is often used when dealing with fuzzy numbers and their membership function. However, according to the form of Eq.(6)-(8), the operation of fuzzy number $\tilde{t}_{ij}$will be too complicated and its membership function is hard to express. Therefore, a numerical method is chosen to obtain proper value of parameters. Similar methods have been proposed by Huang [8] and Jamkhaneh [9]. The process is shown in Figure 1 as below.

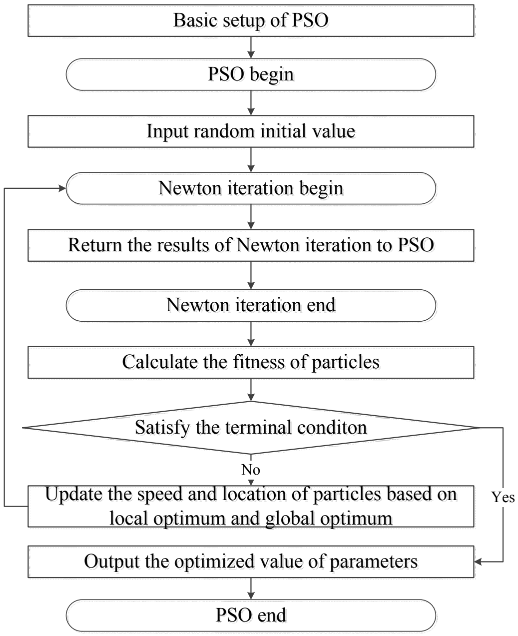

Figure 1-The process of optimization of ALT

Based on the PSO, the process provides the range of the three parameters $\tilde{a}, \tilde{b}$ and $\tilde{m}$ when given a α-cut level of fuzzy failure data $\tilde{t}_{ij}$. The α-cut level restrains the range of $\tilde{t}_{ij}$ which represents the uncertainty of failure data. α equals to 1 means the failure data is precise. When the exact failure time is totally unknown, α is zero.

The core equations of optimization are below. Eq.(13) and Eq.(14) just show the way to calculate parameter $\tilde{a}$. Parameter $\tilde{b}$ and $\tilde{m}$ are calculated in the same way. Results will be obtained after times of iterations combining these equations and PSO.

$$\tilde{a} \in \left[\tilde{a}_\alpha^L, \tilde{a}_\alpha^U\right] = \left\{ solve\left(Eq(6)-(8)\right) \middle| \tilde{t}_{ij} \in C_\alpha\left(\tilde{T}\right)\right\} \tag{13}$$

where

$$\begin{aligned} \tilde{a}_\alpha^L &= \min\left\{ solve\left(Eq(6)-(8)\right) \middle| \tilde{t}_{ij} \in C_\alpha\left(\tilde{T}\right)\right\} \\ \tilde{a}_\alpha^U &= \max\left\{ solve\left(Eq(6)-(8)\right) \middle| \tilde{t}_{ij} \in C_\alpha\left(\tilde{T}\right)\right\} \\ C_\alpha\left(\tilde{T}\right) &= \left[\left(\tilde{T}_i\right)_\alpha^L, \left(\tilde{T}_i\right)_\alpha^U\right] \end{aligned} \tag{14}$$

Fuzzy interval of characteristic life $\tilde{\eta}$ under normal stress level S_0 then will be given through Eq.(15) by invoking PSO a second time .

$$\tilde{\eta} = \exp\left(\tilde{a} + \tilde{b}\varphi(S_0)\right) \tag{15}$$

Fuzzy interval of reliability $\tilde{R}$ will also obtained by PSO.

$$\tilde{R}(t) = \exp\left\{-\left(\frac{t}{\tilde{\eta}}\right)^{\tilde{m}}\right\} \tag{16}$$

4 CASE STUDY

Some details will be discussed combining with a case study. The data is quoted from reference [10]. It is a constant-stress accelerated life testing (CSALT) of some tantalum electrolytic capacitor with four stress level. Temperature is the sensitive stress of samples. In order to estimate the lifetime and reliability of the product under its normal working condition(50℃), such a CSALT is conducted. Test conditions and results have been presented in *Table 1*.

Table 1 Test conditions and results

Test condition and results	**Failure time of samples(unit: hour)**
S_1=85℃,n_1=60, r_1=26	262.52,445.05,500,600,836,850,867.20,1250,1250, 1750,1750,1750,1750,1750,1750,1993,2050.5,2500, 2500,2500,2608.1,2608.1,2608.1,2609.1,3500,3500
S_2=125℃,n_2=60, r_2=33	2,11.3,20,22,50,50,52,52,121.45,126.25,127.4,128.3 143.3,146.0,152,152,162,196.3,199.3,200,205.3, 236.45,285.3,315,315,315,317,400,450.45,493,493, 493,493
S_3=150℃,n_3=20, r_3=12	16.3,26.06,32,36.42,42.06,60.10,83.18,92.1,92.1, 100,120,165
S_4=175℃,n_4=20, r_4=14	15.06,18.18,23.48,23.54,24.12,24.24,40.18,48.36, 48.42,48.48,50,50.12,74.24,114.54

The reference [10] does not provide the inspection interval. As all know, the higher stress-level is, the shorter is the inspection interval. So this paper assumes the value of different intervals as below to illustrate the method. Besides, the taken time of inspection itself can be ignored. The inspection interval of S_1 is 48 hours and expressed as h_1=48. And h2=10, h3=2, h4=0.5. The unit is hour. Based on fuzzy theory and the membership function provided, the data can be transferred into fuzzy interval data. For example, the first data under S1 is 262.52. Given the assumptions above, the inspection interval of S1 is 48 hours. So the first failure is detected in the 6th inspection. The exact timing point is between 240(48*5) and 288. Therefore its membership function can be expressed as

$$\mu_{\tilde{t}_{ij}}\left(t_{ij}\right) = \begin{cases} 1 - \dfrac{262.52 - t_{ij}}{22.52} & 240 \le t_{ij} < 262.52 \\ 1 & t_{ij} = 262.52 \\ 1 - \dfrac{t_{ij} - 262.5}{25.48} & 262.52 < t_{ij} \le 288 \\ 0 & else \end{cases} \tag{17}$$

The fuzzy interval value of parameter $\tilde{a}, \tilde{b}$ and $\tilde{m}$ can be estimated by matlab as below.

$$\begin{aligned} &when \quad \alpha = 0.4 \\ &\tilde{a}_\alpha = \left[\tilde{a}_\alpha^L, \tilde{a}_\alpha^U\right] = [-14.65, -14,89] \\ &\tilde{b}_\alpha = \left[\tilde{b}_\alpha^L, \tilde{b}_\alpha^U\right] = [7315.12, 7361.65] \\ &\tilde{m}_\alpha = \left[\tilde{m}_\alpha^L, \tilde{m}_\alpha^U\right] = [0.952,\ 1.166] \end{aligned} \tag{18}$$

According to Eq.(15) and PSO, fuzzy characteristic life of samples under normal stress can be obtained as

$$\tilde{\eta}_\alpha = \left[\tilde{\eta}_\alpha^L, \tilde{\eta}_\alpha^U\right] = [2423.35, 3433.97] \tag{19}$$

If α becomes bigger, which shows that the data is more precise, the fuzzy interval will be smaller as below.

$$\begin{aligned} &when \quad \alpha = 0.7 \\ &\tilde{a}_\alpha = \left[\tilde{a}_\alpha^L, \tilde{a}_\alpha^U\right] = [-14.7, -14,8] \\ &\tilde{b}_\alpha = \left[\tilde{b}_\alpha^L, \tilde{b}_\alpha^U\right] = [7319.77, 7342.68] \\ &\tilde{m}_\alpha = \left[\tilde{m}_\alpha^L, \tilde{m}_\alpha^U\right] = [0.987,\ 1.096] \end{aligned} \tag{20}$$

According to Eq.(15) and PSO, fuzzy characteristic life of samples under normal stress can be obtained as

$$\tilde{\eta}_\alpha = \left[\tilde{\eta}_\alpha^L, \tilde{\eta}_\alpha^U\right] = [2596.3, 3080.2] \quad (21)$$

The length of interval is shorter which shows a more definite range is obtained.

Assuming the data is precise which means α=1, the result of traditional statistical method based on MLE is as below. The value is included in the fuzzy intervals above.

$$\begin{aligned} &when \quad \alpha=1 \\ &a=-14.718; b=7326.03 \\ &m=1.0598; \eta=2873.3 \end{aligned} \quad (22)$$

The value of α is settled by considering the accuracy of the failure time data. More accurate the data is, α is closer to 1. Test conditions and the differences between inspectors also should be measured. Sometimes expert comments may be required to make a decision.

Therefore, based on the analysis above, the fuzzy reliability should be plotted as curve surface given a pair of value of the characteristic life $\tilde{\eta}$ and the shape parameter $\tilde{m}$ more than a single line. Take the data when α=0.7 as an example. Two pieces of surface restrain the range of the value of fuzzy reliability as time goes.

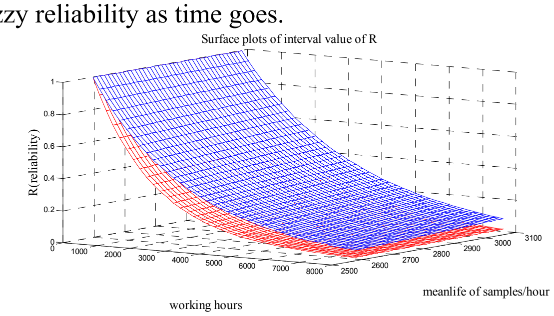

Figure 2- Curve surface of fuzzy reliability

5 CONCLUSIONS AND FUTURE WORK

In this paper, fuzzy theory is first used in ALT combing with MLE in order to solve a kind of uncertainty problem. As the recorded failure data is not precise, the output of ALT should be interval estimation which is more reasonable and credible. Therefore, based on fuzzy theory and maximum likelihood estimation, a numerical method is proposed. This method can also be used in other statistical model of accelerated tests when test data possesses kind of uncertainty.

However, when the statistical model is complicated, the output of Newton Raphson iteration method is very strict with initial value and may be not accurate enough. Besides, the iterations may take too much time. More optimization methods need to be researched to solve the equations of MLE in further study.

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