Optimal design for accelerated degradation tests with stochastic model uncertainty

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- Model uncertainty in ADT planning.
- Bayesian model averaging method.
- Optimization problem.
- Illustrative example & sensitivity analysis.
- Conclusion & future work.

Literature Review

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- Why is Accelerated degradation testing ? – High reliability & Long lifespan VS Time constraint.
- How to design ADT plan ? – Target &
- What is the model ADT data? Degra
- How to quantify model uncertainty
  Gamma process and inverse Gaussian process ?
- Are there any relevant researches on this topic ?

1. Total cost
2. Estimation precision of the lifetime
3. Etc.

1. Sample size allocation
2. Inspection times
3. Stress levels
4. Etc.

Model uncertainty

- **Stochastic process models for degradation modeling**
  - **Wiener process (M1)**
  - **Gamma process (M2)**
  - **Inverse Gaussian process (M3)**

Basic assumptions:

- $\forall t2 > t1 \geq s2 > s1$, $X(t)$ has independent increments, that is, $X(t2) - X(t1)$ and $X(s2) - X(s1)$ are independent.
- $\forall t \geq s > 0, X(t) - X(s)$ follows **certain distributions**, whose mean and variance are proportional to $\Lambda(t) - \Lambda(s)$.

where $\Lambda(t)$ is the time-scale transformation.

Model uncertainty

- **Unified stochastic process model *USP*(*a*, *b*)**

$$E[X] = \mu\Lambda(t) \qquad \mathrm{Var}[X] = \sigma^2\Lambda(t)$$

Model	*a*	*b*>0	PDF
***M*₁**	$\mu\Lambda(t)$	$\sqrt{\sigma^2\Lambda(t)}$	$f_N(x\|a,b) = \frac{1}{\sqrt{2\pi}b}\exp\left[-\frac{(x-a)^2}{2b^2}\right]$
***M*₂**	$\mu^2\Lambda(t)/\sigma^2$	σ^2/μ	$f_{Ga}(x\|a,b) = \frac{b^{-a}}{\Gamma(a)}x^{a-1}\exp\left(-\frac{x}{b}\right), x > 0$
***M*₃**	$\mu\Lambda(t)$	$\mu^3\Lambda^2(t)/\sigma^2$	$f_{IG}(x\|a,b) = \sqrt{\frac{b}{2\pi x^3}}\exp\left[-\frac{b(x-a)^2}{2a^2x}\right], x > 0$

Model uncertainty

❑ Acceleration model

$$\mu_i = exp(\alpha_0 + \alpha_1 s_i)$$

where $s_i \epsilon [0,1]$ is the normalized accelerated stress level by

$$s_i = \begin{cases} \dfrac{1/s'_0 - 1/s'_i}{1/s'_0 - 1/s'_H} & Arrhenius\ relation \\ \dfrac{lns'_i - lns'_0}{lns'_H - lns'_0} & Power\ law\ relation \\ \dfrac{s'_i - s'_0}{s'_H - s'_0} & Exponential\ relation \end{cases}$$

The degradation rate at the use condition is

$\mu_0 = exp(\alpha_0)$.

where s'_0 and s'_H are the normal and highest stress levels, i = 1,2,…,K.

Model uncertainty

The influence of Model Uncertainty

- **The p-th quantile lifetime at the use condition**

$$t_p = \Lambda^{-1}\left[\frac{\beta}{4}\left(\alpha z_p + \sqrt{4 + \alpha^2 z_p^2}\right)^2\right]$$

where z_p is the p-quantile of the standard normal distribution, and $\Lambda^{-1}(\cdot)$ is the inverse function of $\Lambda(\cdot)$.

Based on the asymptotically normal distribution, the asymptotic variance of *tp* can be given as

$$Avar(t_p) = (\nabla t_p)' I^{-1}(\theta) \nabla t_p$$

where *I*(*θ*) is the expected Fisher information matrix.

Bayesian model averaging (BMA)

Uncertainty quantification

Considered that Δ is the quantity of interest, i.e. t_p at the use condition, its posterior distribution given data D is

$$f(\Delta|D) = \sum_{c=1}^{C} f(\Delta|M_c, D) Pr\{M_c|D\}$$

where $f(\Delta|M_c, D)$ is the posterior density of Δ assuming that M_c is the true model, $Pr\{M_c|D\}$ is the posterior probability of model M_c.

$$Pr\{M_c|D\} \propto f(D|M_c) Pr\{M_c\}$$

where $Pr(M_c)$ is the prior probability of model M_c, $f(D|M_c)$ is the integrated likelihood of model M_c

$$f(D|M_c) = \int f(D|\boldsymbol{\theta}_c, M_c) f(\boldsymbol{\theta}_c|M_c)$$

where $f(D|\boldsymbol{\theta}_c, M_c)$ is the likelihood function under model M_c.

Bayesian model averaging (BMA)

- **Uncertainty quantification**

For Δ, its posterior mean is

$$E(\Delta|D) = \sum_{c=1}^{C} \hat{\Delta}_c \mathrm{Pr}\{M_c|D\}$$

and its variance is

$$Var(\Delta|D) = \sum_{c=1}^{C} \left(Var(\Delta|M_c, D) + \hat{\Delta}_c\right) \mathrm{P}r\{M_c|D\} - E(\Delta|D)^2$$

where $\hat{\Delta}_c = E(\Delta|M_c, D)$.

Optimization problem

Single model

$$
\begin{aligned}
\text{Minimize} \quad & Avar(\hat{t}_p) \\
\text{subject to} \quad & 0 \le s_i \le 1,\ s_{i-1} \le s_i,\ i \in [1, K] \\
& \sum_{i=1}^{K} n_i = N,\ n_i \le n_{i-1},\ i \in [2, K] \\
& \sum_{i=1}^{K} m_i = M,\ m_i \le m_{i-1} \\
& n_i\ and\ m_i \in N^+
\end{aligned}
$$

Multiple models

$$
\begin{aligned}
\text{Minimize} \quad & Var[\hat{t}_p | D] \\
\text{subject to} \quad & 0 \le s_i \le 1,\ s_{i-1} \le s_i,\ i \in [1, K] \\
& \sum_{i=1}^{K} n_i = N,\ n_i \le n_{i-1},\ i \in [2, K] \\
& \sum_{i=1}^{K} m_i = M,\ m_i \le m_{i-1} \\
& n_i\ and\ m_i \in N^+
\end{aligned}
$$

Illustrative example

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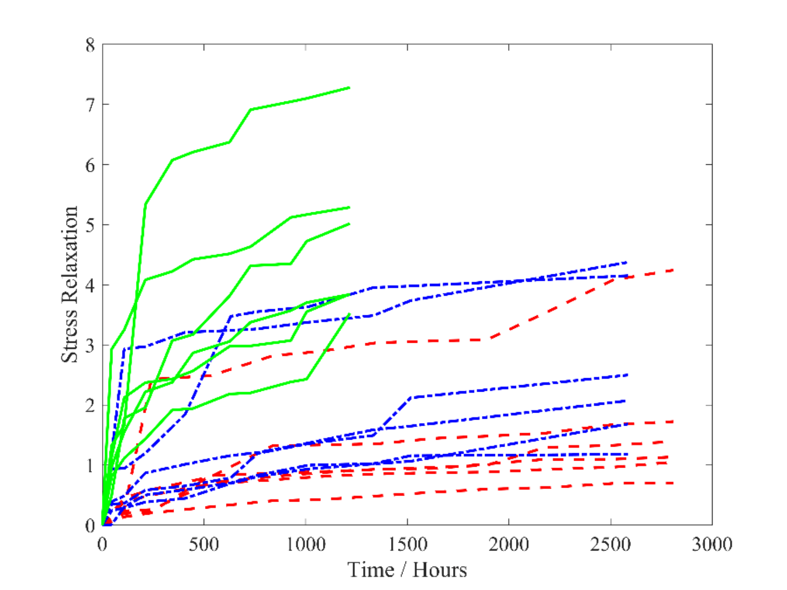

Stress relaxation CSADT data

Contens	Values
Accelerated stresses	65, 85, 100℃
Normal condition	40℃
Failure threshold	30%

Illustrative example

❑ Model selection with Lmax value

Table 2: Parameter estimation for single model.

	Acceleration		Degradation		
Models	α_0	α_1	σ	γ	l_{max}
M_1	-3.4857	2.8951	0.4222	0.2997	-74.6
M_2	-2.9438	1.0417	0.1922	0.4381	55.1
M_3	-3.6858	1.2253	0.2606	0.5181	54.8

- this process maybe ignored in real applications with the assumption of the degradation model, or
- this selection maybe inappropriate for designing ADT plans without the consideration of model uncertainty.

Illustrative example

❑ Model comparison for the 2-level CSADT plan

Assumptions:
- ✓ N = 10 samples
- ✓ M = 100 inspection
- ✓ Time interval 24 hours
- ✓ p = 0.1

Table 3: Optimal plan for 2-level CSADT.

	Samples		Inspection		Stress		
Models	n_1	n_2	m_1	m_2	s_1	s_2	$std(\hat{t}_p)$
M_1	8	2	96	4	0.59	1	3.32e9
M_2	7	3	84	16	0.59	1	7.44e5
M_3	5	5	67	33	0.73	1	2.54e5
BMA	7	3	82	18	0.61	1	7.16e5

$Pr(M_1 \mid D) = 0$
$Pr(M_2 \mid D) = 0.5628$
&
$Pr(M_3 \mid D) = 0.4372$

- ➢ the optimal second stress level is the same for three candidate model (s2 = 1, i.e.100◦C).
- ➢ the sample allocations, inspection times and the first stress level are different with different choice of the degradation model.
- ➢ M2 is treated as the most suitable model from Table 2 but without the highest prediction precision.

Sensitivity analysis

- **Variation of parameters**

Setting true values as a0 = 2, a1 = 1.5, σ = 0.5, γ = 0.4 and the variations into three levels, i.e. +10%, 0, -10%. *Pr*(*Mc* | *D*) = 1/3.

Table 4: Sensitivity analysis of the variation of parameters on the optimal 2-level CSADT plan.

ϵ_1	ϵ_2	ϵ_3	ϵ_4	n_1	n_2	m_1	m_2	s_1	s_2	$std(\hat{t}_p)$
+10%	+10%	+10%	+10%	7	3	80	20	0.46	1	7.71e4
+10%	0	0	0	7	3	84	16	0.45	1	3.59e5
+10%	-10%	-10%	-10%	8	2	88	12	0.41	1	2.22e6
0	10%	-10%	0	7	3	81	19	0.48	1	1.90e5
0	0	+10%	-10%	7	3	85	15	0.46	1	1.06e6
0	-10%	0	10%	7	3	82	18	0.40	1	5.11e4
-10%	+10%	0	-10%	7	3	82	18	0.50	1	5.41e5
-10%	0	-10%	+10%	7	3	79	21	0.42	1	2.72e4
-10%	-10%	+10%	0	7	3	83	17	0.42	1	1.16e5

Sensitivity analysis

- **Variation of model posterior probabilities**

Setting true values as a0 = 2, a1 = 1.5, σ = 0.5, γ = 0.4
and the variations into three levels, i.e. +10%, 0, -10%,

Table 5: Sensitivity analysis of the variation of model posterior probabilities on the optimal 2-level CSADT plan.

$Pr(M_1\|D)$	$Pr(M_2\|D)$	$Pr(M_3\|D)$	n_1	n_2	m_1	m_2	s_1	s_2	$std(\hat{t}_p)$
1/6	1/3	1/2	6	4	75	25	0.49	1	1.86e5
1/6	1/2	1/3	7	3	79	21	0.51	1	1.90e5
1/3	1/6	1/2	7	3	83	17	0.45	1	1.97e5
1/3	1/2	1/6	7	3	83	17	0.45	1	2.03e5
1/2	1/6	1/3	8	2	88	12	0.39	1	2.07e5
1/2	1/3	1/6	8	2	88	12	0.39	1	2.10e5

Summary

- The BMA method is introduced to analyze the problem of stochastic model uncertainty on designing the optimal ADT plan.
- The simulated stress relaxation CSADT data shows that each stochastic degradation model can produce different optimal plan and prediction precision.
- The sensitivity study shows that the optimal plan is less sensitive to the variation of parameters but to model posterior probabilities, while that for the prediction precision is reverse.

Future Work

- The study of the random effects due to the variation of the tested samples
- Maybe the measurement error
- Compative study with reference model

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