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Optimal design for accelerated degradation tests with stochastic model uncertainty

Le Liu, Xiao-Yang Li, Tong-Min Jiang & Jian-Guo Zhang
School of Reliability and Systems Engineering, Beihang University, Beijing, China
Science and Technology on Reliability and Environmental Engineering Laboratory, Beijing, China

ABSTRACT: In this work, we address the issue of stochastic model uncertainty on the optimal design for accelerated degradation tests. All three candidate models, i.e. Wiener, Gamma and inverse Gaussian, are considered in the degradation modelling and averaged by the posterior model probability with the Bayesian model averaging method. The averaged model is proposed to integrate the contributions from each model. After that, the objective function is constructed to minimize the posterior asymptotic variance of the estimate of the $p-$th quantile lifetime at the use conditions, where the sample size allocation, inspection times and stress levels are treated as the decision variables. In the illustrative example, the simulated stress relaxation data under three temperature stress levels is used to analyze the effect of model uncertainty on the optimal plan. Sensitivity study is also given to robustness of the optimal plan on the variation of parameters and model posterior probabilities.

1 INTRODUCTION

Accelerated degradation testing (ADT) is generally used for lifetime and reliability evaluation for highly reliable products that no failure data are available in traditional reliability tests (Nelson 1990). In order to effectively conduct ADT with the limited resources, special attention is given to the total costs (Tang et al. 2004), the estimation precision of the lifetime (e.g. the asymptotic variance of the $p-$quantile, see Liao et al. (2006) and Pan et al. (2009)), or the approximate variance of the estimated mean time to failure (MTTF) (Tseng et al. 2009), etc. For designing the optimal ADT plan, the aforementioned factors will be chosen as the target with the constraints of sample size or the total costs that of interest, to compute the optimal plan consisting of sample size allocation, inspection times and stress levels, and also ensure the precision of the reliability evaluation results.

For the sake of designing ADT plan, degradation models should be selected through either degradation-path (Yu 2003) or statistical data-driven (Ye and Xie 2015). With the time-dependent structure which is suitable for describing the temporal variation of the degradation process, stochastic process models are widely used for degradation modelling, i.e. Wiener process, Gamma process and inverse Gaussian process.

Lim & Yum (2011) assumed that the degradation paths follow the Wiener process, and obtained the optimal constant stress ADT (CSADT) plan with the stress levels and the allocated tested samples through minimizing the $q-$quantile of the lifetime at the use conditions. Then, Hu et al. (2015) studied the optimal plan design for step stress ADT (SSADT) under the Wiener degradation assumption. Tseng et al. (2009) considered the situation that the degradation paths are monotonous, especially for the fatigue degradation process, and selected Gamma process as the degradation model to design the optimal SSADT through minimizing the approximated variance of the estimated MTTF with the constraint of total cost, while the decision variables are sample size, inspection frequency and time. Moreover, Tsai et al. (2012) considered the random effect due to the variation of samples to conduct optimal plan design based on Gamma process. Recently, Ye et al. (2014) performed the optimal plan design based on inverse Gaussian process model which has more flexible structures to capture the uncertainties, e.g. random drifts model, random volatility model and random drift-volatility model, see Wang and Xu (2010), Peng et al. (2014), Ye and Chen (2014) and Peng (2015).

In traditional design of ADT plan, the degradation model is assumed without considering the problem of model uncertainty, which has been shown that it will lead to significantly different reliability and lifetime evaluation results, see Yu and Chang (2012) and Liu et al. (2016). To account for such uncertainty, Zio and Apostolakis (1996) introduced two methods which are adjustment factor that nominates one reference model and updates it with new data, and model averaging to mixture all the candidate models. For the adjustment factor method, Tseng and Lee (2016) proposed a class of exponential-dispersion degradation models to complete the optimum allocation for 2-level and 3-level ADT, which can capture all three candidate models as its limiting cases with different choices of parameter d. While in this study, we will analyze the effect of model uncertainty on optimal plan design through Bayesian model averaging (BMA) method to integrate the contributions of each candidate model.

This paper is organized as follows: Section 2 introduces the methodology for ADT planing through BMA. Simulation study is performed to illustrate the proposed method in Section 3. Sensitivity study is given to the variations of parameters and model posterior probabilities on the optimal plan in Section 4. Section 5 concludes this paper.

2 ADT PLANNING THROUGH BAYESIAN MODEL AVERAGING

2.1 *Degradation and acceleration models*

Let us denote the three candidate models by $M_c, c = 1, 2, 3$ for Wiener, Gamma and inverse Gaussian processes. In general, we define the degradation path $X(t)$ as

- $\forall t_2 > t_1 \geq s_2 > s_1, X(t)$ has independent increments, that is, $X(t_2) - X(t_1)$ and $X(s_2) - X(s_1)$ are independent.
- $\forall t \geq s > 0, X(t) - X(s)$ follows normal (M_1), Gamma (M_2) and inverse Gaussian (M_3) distributions, whose mean and variance are proportional to $\Lambda(t) - \Lambda(s)$.

To present the stochastic degradation model uncertainty, the mean and variance for $X(t)$ are given as

$$E(x) = \mu\Lambda(t) \qquad Var(x) = \sigma^2\Lambda(t) \tag{1}$$

where $\Lambda(t)$ is the time-transformation.

For simplicity, let $USP(a, b)$ denote the unified stochastic model with specific definitions of a and b for all candidate models. Table 1 gives the corresponding parameters for M_c. For acceleration model which shows the relationship among different stress levels, the log-linear formula is generally used as

$$\mu_i = exp(\alpha_0 + \alpha_1 s_i) \tag{2}$$

where α_0 and α_1 are two unknown parameters, and $s_i \in [0, 1], i = 1, 2 \ldots K$ is the normalized stress level after (Lim and Yum 2011).

Moreover, the $p-$quantile lifetime at the use conditions for $USP(a, b)$ is (Liu et al. 2016)

$$t_p = \Lambda^{-1}\left[\frac{\beta}{4}\left(\alpha z_p + \sqrt{4 + \alpha^2 z_p^2}\right)^2\right] \tag{3}$$

where $\alpha = \frac{\sigma}{\sqrt{\omega\mu}}$, $\beta = \frac{\omega}{\mu}$, z_p is the $p-$quantile of the standard normal distribution and $\Lambda^{-1}(\cdot)$ is the inverse function of $\Lambda(\cdot)$. Based on the asymptotically normal distribution, the asymptotic variance of t_p can be given as

$$AVar(t_p) = (\nabla t_p)' \boldsymbol{I}^{-1}(\boldsymbol{\theta}) \nabla t_p \tag{4}$$

where $\mathbf{I}(\boldsymbol{\theta})$ is the expected Fisher information matrix. To guarantee the prediction precision of the quantile of lifetime distribution at the use conditions, efforts have been devoted to minimize (4) with the assumed degradation model, see Yu (2003), Liao et al. (2006), Lim and Yum (2011) and Ye et al. (2014).

2.2 *Bayesian model averaging*

The problem of model uncertainty can be intuitively solved through Bayesian model averaging method with the contribution of each model averaged by the model posterior probabilities (Hoeting et al. 1999).

Assuming that Δ is the quantity of interest, it posterior distribution with the given data D can be presented as

$$Pr(\Delta|D) = \sum_{c=1}^{C} Pr(\Delta|M_c, D) Pr(M_c|D) \tag{5}$$

where $Pr(M_c|D)$ is the posterior probability of M_c given data D, that is

$$Pr(M_c|D) = \frac{Pr(D|M_c)Pr(M_c)}{\sum_{l=1}^{C} Pr(D|M_l)Pr(M_l)} \tag{6}$$

where

$$Pr(D|M_c) = \int Pr(D|\theta_c, M_c) Pr(\theta_c|M_c) d\theta_c \tag{7}$$

is the integrated likelihood of model M_c.

For Δ, its posterior mean is

$$E[\Delta|D] = \sum_{c=1}^{C} \hat{\Delta}_c Pr(M_c|D) \tag{8}$$

and its variance is

$$\begin{aligned} Var[\Delta|D] \;&=\; \sum_{c=1}^{C} (Var[\Delta|M_c, D] + \hat{\Delta}_c^2) Pr(M_c|D) \\ &-\; E[\Delta|D]^2 \end{aligned} \tag{9}$$

where $\hat{\Delta}_c = E[\Delta|M_c, D]$.

2.3 *Optimization problem*

For single model M_c, we can simply take (4) as the objective function with the constraints of sample sizes

Table 1. Parameter definition for three candidate models.

Models	Wiener process		Gamma process		Inverse Gaussian process	
	Mean	STD	Shape	Scale	Mean	Shape
Parameters	$\mu\Lambda(t)$ a	$\sqrt{\sigma^2\Lambda(t)}$ b	$\frac{\mu^2\Lambda(t)}{\sigma^2}$ a	$\frac{\sigma^2}{\mu}$ b	$\mu\Lambda(t)$ a	$\frac{\mu^3\Lambda^2(t)}{\sigma^2}$ b

and time to find the optimal sample allocation, inspection times and stress levels. The optimization problem can be organized as

$$\begin{aligned}
\text{Minimize} \quad & Avar(\hat{t}_p) \\
\text{subject to} \quad & 0 \le s_i \le 1,\ s_{i-1} \le s_i,\ i \in [1, K] \\
& \sum_{i=1}^{K} n_i = N,\ n_i \le n_{i-1},\ i \in [2, K] \\
& \sum_{i=1}^{K} m_i = M,\ m_i \le m_{i-1} \\
& n_i \ and\ m_i \in N^+
\end{aligned} \tag{10}$$

When accounting for model uncertainty, the objective function should be the posterior variance of $\hat{t}_p$ with (9) where $\Delta = t_p$. To simplify the calculation of $Var[\hat{t}_p|D]$, we replaced $Var[\hat{t}_p|M_c, D]$ in (9) by $Avar(\hat{t}_p^c)$ in (4) for each candidate model. Both of this two formulas present the variance of the $p-$quantile lifetime at the use conditions but that one is from the MLE analysis, while the other one is from Bayesian analysis which is time-consuming.

In this way, we can at first compute optimal plan for model M_c and $Avar(\hat{t}_p^c)$ with the ADT data D, then compute the model posterior probability $Pr(M_c|D)$ through BMA analysis, and finally produce the $Var[\hat{t}_p|D]$, which is the new objective function capturing the model uncertainty, that is

$$\begin{aligned}
\text{Minimize} \quad & Var[\hat{t}_p|D] \\
\text{subject to} \quad & 0 \le s_i \le 1,\ s_{i-1} \le s_i,\ i \in [1, K] \\
& \sum_{i=1}^{K} n_i = N,\ n_i \le n_{i-1},\ i \in [2, K] \\
& \sum_{i=1}^{K} m_i = M,\ m_i \le m_{i-1} \\
& n_i \ and\ m_i \in N^+
\end{aligned} \tag{11}$$

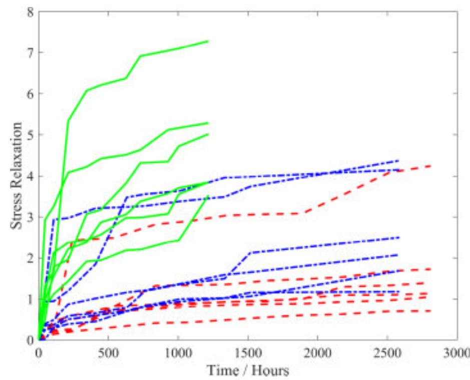

Figure 1. The simulated stress relaxation CSADT data under three temperature stress levels.

2.4 *Statistical inferences*

Suppose that $X(t_{ijk})$ is the k^{th} degradation value of unit j under the i^{th} stress level at time t_{ijk}, $i = 1, 2, \dots, K, j = 1, 2, \dots, n_i, k = 1, 2, \dots, m_i$, where K is the number of stress levels, n_i is the number of test samples under the i^{th} stress level and m_i is the number of measurements. Given that $x_{ijk} = X(t_{ijk}) - X(t_{ij(k-1)})$ is the degradation increment for CSADT with the time increment $\Lambda_{ijk} = \Lambda(t_{ijk}) - \Lambda(t_{ij(k-1)})$, the likelihood function of ADT data D is

$$L(D|\boldsymbol{\theta}) = \prod_{i=1}^{K}\prod_{j=1}^{n_i}\prod_{k=1}^{m_{ij}} f_{USP}(x_{ijk}|a_{ijk}, b_{ijk}) \tag{12}$$

The estimates of $\boldsymbol{\theta} = [\alpha_0, \alpha_1, \sigma, \gamma]$ can be obtained by maximizing its logarithm formulation, $l(\boldsymbol{\theta}_c|M_c, D)$, with the definitions in Table 1.

3 ILLUSTRATIVE EXAMPLE

The stress relaxation data is simulated in order to illustrate the problem of stochastic degradation model uncertainty on designing optimal ADT plan. Figure 1 shows the degradation paths for 18 samples under three temperature stress levels (six at each level), i.e. 65, 85, 100 $^\circ C$ with the settings in Yang (2007). Note that the electrical connector will fail when its stress relaxation comes over 30%, that is to say, $\omega = 30$. The normal condition s_0 is 40°C.

Table 2. Parameter estimation for single model.

Models	Acceleration		Degradation		
	α_0	α_1	σ	γ	l_{max}
M_1	–3.4857	2.8951	0.4222	0.2997	–74.6
M_2	–2.9438	1.0417	0.1922	0.4381	55.1
M_3	–3.6858	1.2253	0.2606	0.5181	54.8

Table 3. Optimal plan for 2-level CSADT.

Models	Samples		Inspection		Stress		
	n_1	n_2	m_1	m_2	s_1	s_2	$std(\hat{t}_p)$
M_1	8	2	96	4	0.59	1	3.32e9
M_2	7	3	84	16	0.59	1	7.44e5
M_3	5	5	67	33	0.73	1	2.54e5
BMA	7	3	82	18	0.61	1	7.16e5

The simulated stress relaxation CSADT data under three temperature stress levels.

With each stochastic degradation model, the unknown parameters can be estimated through the statistical method in Section 2.4. The results are listed in Table 2. Given from the l_{max} from Table 2, Gamma process model is the most suitable one with the largest value, which can be further used for designing optimal ADT plan. However, this process maybe ignored in real applications with the assumption of the degradation model, or this selection maybe unappropriate for designing ADT plans without the consideration of model uncertainty.

Assuming that we have $N = 10$ available samples and $M = 100$ inspection times (time interval is twenty-four hours), ADT planning can be conducted using (10) with the parameters in Table 2 fixed, and the $p = 0.1$ quantile of the lifetime at the use conditions is that of interest. Table 3 shows the optimal two-level CSADT plans for single model.

Given from Table 3, the optimal second stress level is the same for three candidate model ($s_2 = 1$, i.e. 100°C). However, the sample allocations, inspection times and the first stress level are different with different choice of the degradation model. For instance, 8/2 samples and 96/4 inspections with $s_1 = 0.59$ will be given for the ADT plan if M_1 is chosen, while 7/3 and 84/16 with 0.59 for M_2, and 5/5 and 67/33 with 0.73 for M_3. Hence, significant different plans will be proposed for different selections of the degradation models, which means the stochastic degradation model uncertainty has effect on designing the optimal plan. Also, M_2 is treated as the most suitable model from Table 2 but without the highest prediction precision.

The BMA method given in Section 2.2 is used to accounting for such uncertainty. The results show that the posterior model probabilities for each model are $Pr(M_1|D) = 0$, $Pr(M_2|D) = 0.5628$ and $Pr(M_3|D) = 0.4372$, respectively.

The new optimal plan can be obtained from (11), which is listed at the end of Table 3. For BMA model, the optimal plan is consistent with M_2 for the sample allocations ($n_1 = 7, n_2 = 3$), but has the compromised inspection times ($m_1 = 82, m_2 = 18$) and the first stress level ($s_1 = 0.61$, i.e. 74°C) with higher precision ($7.16e5 < 7.44e5$) through utilizing the prediction capacity of M_3.

Therefore, the BMA model can not only account for model uncertainty, but also integrate all the advantages from each candidate model, which is useful for designing ADT plan in real applications with no such knowledge about the selection of degradation model.

4 SENSITIVITY ANALYSIS

In this section, both the influences of parameters and model posterior probabilities on the optimal 2-level CSADT plan will be studied to observe the robustness of the optimal plan.

4.1 *The variation of parameters*

The optimal plan depends on the true values of the unknown parameters $\boldsymbol{\theta}$. In order to analyze the robustness of the BMA model which accounts for the contributions from three stochastic degradation models, we set the true values as $\alpha_0 = -2, \alpha_1 = 1.5, \sigma = 0.5, \gamma = 0.4$ and the variations into three levels, i.e. $+10\%, 0, -10\%$, given in the first four columns of Table 4 according to a $L_9(3^4)$ orthogonal array. Meanwhile, the contributions are treated as equal for each model, i.e. $Pr(M_c|D) = 1/3$. With the same settings about the CSADT in Section 3, the results are listed at the right of Table 4.

Given from the results, the optimal plan is quite robust with slightly change in samples allocation (only the third one has one more sample in s_1), inspections times (the deviation is nine), and stress level one s_1 (the deviation is 0.10, from 61.5°C to 67.4°C). It is interesting to see that the variance of the $p = 0.1$ quantile lifetime varies from 2.72e4 to 2.22e6. Hence, the prediction precision is effected by the variation of parameters.

4.2 *The variation of model posterior probabilities*

When accounting for the model uncertainty, the optimal plan also depends on the model posterior probabilities which present the contributions of each model. Hence, this section will study this effect under three levels for each model, see the first three columns of Table 5. The settings for parameters and ADT plans are the same with that in Section 4.1.

Given from the results, the optimal plan varies with different model posterior probabilities, especially when that for M_1 increases, and more samples and

Table 4. Sensitivity analysis of the variation of parameters on the optimal 2-level CSADT plan.

ϵ_1	ϵ_2	ϵ_3	ϵ_4	n_1	n_2	m_1	m_2	s_1	s_2	$std(\hat{t}_p)$
+10%	+10%	+10%	+10%	7	3	80	20	0.46	1	7.71e4
+10%	0	0	0	7	3	84	16	0.45	1	3.59e5
+10%	-10%	-10%	-10%	8	2	88	12	0.41	1	2.22e6
0	10%	-10%	0	7	3	81	19	0.48	1	1.90e5
0	0	+10%	-10%	7	3	85	15	0.46	1	1.06e6
0	-10%	0	10%	7	3	82	18	0.40	1	5.11e4
-10%	+10%	0	-10%	7	3	82	18	0.50	1	5.41e5
-10%	0	-10%	+10%	7	3	79	21	0.42	1	2.72e4
-10%	-10%	+10%	0	7	3	83	17	0.42	1	1.16e5

Table 5. Sensitivity analysis of the variation of model posterior probabilities on the optimal 2-level CSADT plan.

$Pr(M_1\|D)$	$Pr(M_2\|D)$	$Pr(M_3\|D)$	n_1	n_2	m_1	m_2	s_1	s_2	$std(\hat{t}_p)$
1/6	1/3	1/2	6	4	75	25	0.49	1	1.86e5
1/6	1/2	1/3	7	3	79	21	0.51	1	1.90e5
1/3	1/6	1/2	7	3	83	17	0.45	1	1.97e5
1/3	1/2	1/6	7	3	83	17	0.45	1	2.03e5
1/2	1/6	1/3	8	2	88	12	0.39	1	2.07e5
1/2	1/3	1/6	8	2	88	12	0.39	1	2.10e5

inspection times are given in s_1. However, the prediction accuracy remains stable with a minor change.

5 CONCLUSIONS

In this work, the BMA method is introduced to analyze the problem of stochastic model uncertainty on designing the optimal ADT plan. Through the model posterior probabilities, the contributions of each candidate model are utilized, and the posterior variance of the $p-$quantile lifetime at the use conditions is treated as the new objective function to find the best allocations of samples, inspection times and stress levels. The simulated stress relaxation CSADT data shows that each stochastic degradation model can produce different optimal plan and prediction precision. Meanwhile, the sensitivity study shows that the optimal plan is less sensitive to the variation of parameters but to model posterior probabilities, while that for the prediction precision is reverse. Further research could be given to the study of the random effects due to the variation of the tested samples.

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