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Nonlinear accelerated degradation analysis based on the general Wiener process

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ABSTRACT: Accelerated Degradation Testing (ADT) is an efficient technique for lifetime evaluation of high reliable products whose inner deterioration can be traced by an outer performance characteristic with time. With great physical and stochastic properties, the linear Wiener process or its time transformation is widely used for acquiring lifetime information from the ADT data. However, this method is not applicable for the situation where the degradation path cannot be linearized. Thus, in this work, we address the issue of nonlinear ADT analysis through the general Wiener process, which can deal with not only the nonlinear degradation paths that cannot be linearized, but also the linear ones as its limiting cases. The Akaike Information Criterion (AIC) is introduced to compare the models. The LED case study demonstrates the effectiveness of the general Wiener process model over the common used time-scale transformation Wiener process model on nonlinear ADT analysis.

1 INTRODUCTION

For the lifetime and reliability evaluation of high reliable products, Accelerated Degradation Testing (ADT) is widely used to accomplish such task with hasher than normal test conditions (Meeker et al. 1998). During the past decades, ADT has been applied into many application, e.g. battery (Jung et al. 2014), rubber component (Woo et al. 2010), light bars (Wang and Chu 2012), etc. In general, the analysis of ADT data is to model the degradation path and derive the failure time when the degradation paths exceed the failure threshold, then extrapolate the failure time at high stress conditions to normal condition. The stochastic processes are the commonly used methods for ADT modeling which have excellent physical properties and mathematical attraction, like Wiener process, Gamma process and inverse-Gaussian process (Ye and Xie 2015). Due to the form of normal distribution, the linear Wiener process is the most preferable model in ADT field.

Li & Jiang (2009) studied the optimum plan design of constant stress ADT (SSADT) with competing failure modes through a linear Winer process model. While, Lim & Yum (2011) used the linear Wiener process as the degradation model to study the optimal design of Constant Stress ADT (CSADT) plan. Pan & Balakrishnan (2010) considered the situation when the time-point of elevating stress levels is random variable in SSADT, and selected the linear Wiener and Gamma processes as the degradation models. Ho (2012) utilized the linear Wiener process model for ADT analysis, which is to describe the influence of product material and random effects from the environment. In real applications, Wang et al. (2013) proposed an integrated Bayesian reliability evaluation method with both ADT and field information for super luminescent diode, where a linear Wiener process is used as the degradation model.

From above research, it is known that the linear Wiener process is commonly used in ADT analysis. However, there exists nonlinear degradation processes due to the inner response of product performance for outer environmental conditions. It is improper to model the nonlinear degradation path with linear Wiener process model. Hence, an alternative way is to give a time-scale transformation.

Whitmore & Schenkelberg (1997) is the first to realize this idea with the time-scale transformation $t = 1 - exp(-\lambda r^{\gamma})$ and $t = r^{\lambda}$ in linear Wiener process model for nonlinear ADT analysis. In recent work, Tang et al. (2014) used a time-scale transformation by replacing the t in linear Wiener process to $\Lambda(t)$ to solve the problem of nonlinear ADT analysis.

The problem of time-scale transformation is that there exists a underlying assumption which the linear degradation path can be linearized. However, it is not true for all nonlinear degradation scenarios. In traditional degradation modeling, Wang et al. (2014) proposed a general Wiener process model which can used for both linear and nonlinear degradation modeling with different combinations of $\Lambda(t;\theta)$ and $\tau(t,\gamma)$. Thus, in this paper, the general Wiener process is introduced to solve the problem of nonlinear ADT analysis.

2 ACCELERATED DEGRADATION AND ACCELERATION MODEL

2.1 *Motivation example*

The CSADT data for the LEDs under two stress conditions is shown in Figure 1. Obviously, the degradation of this kind of product experiences a nonlinear path. For more details about the test data, readers are referred to Chaluvadi (2008). In the research of Tang et al. (2014), the dataset is analyzed based on the time-scale transformation Wiener process model, i.e.

$$M_1 : X(t) = \mu\Lambda(t) + \sigma B(\Lambda(t)), \tag{1}$$

where $X(t)$ is a degradation value at time t; whereas μ and σ are draft and diffusion coefficients; $\Lambda(t)$ is the time-scale transformation which can describe the nonlinear property of the degradation process.

The results from Tang et al. (2014) indicates that the unit-to-unit variation has little effect on the estimation of the Mean Time To Failure (MTTF) that of interest. Thus, the random effect is not considered in this paper for simplicity. In addition, the 95% confidence interval of the MTTF at the normal stress under 25 mA is [1848, 57202] hours. However, the estimate of the MTTF from the original paper is about 1346 hours, which means that the time-scale transformation model leads to the significantly overestimated lifetime evaluation results. Hence, a general model will be used and compared with it to illustrate their insights on ADT analysis.

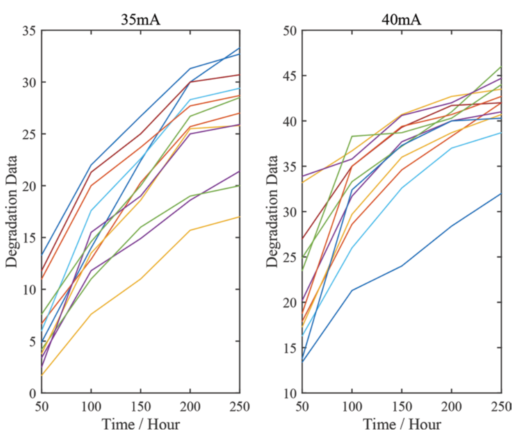

Figure 1. Nonlinear degradation data of 24 LEDs in two stress conditions.

2.2 *Models*

The general Wiener process for degradation modeling is

$$M_0 : X(t) = \mu\Lambda(t;\theta) + \sigma B(\tau(t;\gamma)), \tag{2}$$

where θ and γ are the generalized parameters. Clearly, Equation (1) is a limiting case of Equation (2) when $\Lambda(t;\theta) = \tau(t;\gamma) = \Lambda(t)$. For clarity, Equation (2) is named model M_0, while Equation (1) is model M_1.

In order to extrapolate the lifetime and reliability evaluation results from high stress conditions to normal condition, acceleration model is needed. Specifically, we assume that the drift coefficient μ follows an inverse power relationship with the accelerated stress, current I in the LED case, i.e.

$$\mu = \eta_0 I^{\eta_1}, \tag{3}$$

where η_0 and η_1 are constant numbers.

The lifetime and reliability evaluation can be obtained when the degradation path $X(t)$ firstly exceeds the failure threshold ω, i.e. the First Passage Time (FPT). According to the property of linear Wiener process, the Probability Distribution Function (PDF) of FPT follows an inverse Gaussian distribution (Chhikara 1988), i.e.

$$f(t) = \frac{\omega}{t\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(\omega-\mu t)^2}{2\sigma^2 t}\right), \tag{4}$$

Hence, for the time-scale transformation model M_1, the PDF of FPT can be easily computed through Equation (4).

$$f_1(t) = \frac{\omega}{\Lambda(t)\sqrt{2\pi\sigma^2\Lambda(t)}} \exp\left(-\frac{(\omega-\mu\Lambda(t))^2}{2\sigma^2\Lambda(t)}\right), \tag{5}$$

However, the analytical formula of PDF of FPT for model M_0 cannot be directly obtained from above-mentioned equations. The following section will address this issue.

2.3 *Derivation of failure time distribution*

Let a time transformation is $s = \tau(t;\gamma)$, thus $t = \tau^{-1}(s;\gamma)$. We define $\Lambda(t;\theta) = \Lambda(\tau^{-1}(s;\gamma);\theta) = \rho(s;\theta)$. So, Equation (2) becomes (Wang et al. 2014)

$$X(s) = \mu\rho(s;\theta) + \sigma B(s), \tag{6}$$

and its drift coefficient is

$$\kappa(s;\theta) = \mu\frac{d\rho(s;\theta)}{ds}, \tag{7}$$

Under some mild assumptions, the PDF of FPT for the new degradation process $X(s)$ is (see Therem 2 in Si et al. (2012))

$$p_0(s) = \frac{1}{\sqrt{2\pi s}}\left(\frac{\omega - \mu\rho(s;\theta)}{\sigma s} + \frac{\kappa(s;\theta)}{\sigma}\right) \cdot \exp\left(-\frac{(\omega - \mu\rho(s;\theta))^2}{2\sigma^2 s}\right), \tag{8}$$

Hence, the PDF of FPT for model M_0 can be given by substituting $s = \tau(t;\gamma)$, that is

$$p_0(t) = \frac{1}{\sqrt{2\pi\tau(t;\gamma)}}\left(\frac{\omega - \mu\Lambda(t;\theta)}{\sigma\tau(t;\gamma)} + \frac{\kappa(\tau(t;\gamma);\theta)}{\sigma}\right) \cdot \exp\left(-\frac{(\omega - \mu\Lambda(t;\theta))^2}{2\sigma^2\tau(t;\gamma)}\right)\frac{d\tau(t;\gamma)}{dt}, \tag{9}$$

Herein, the $\int p_0(u)du = 1$ should be satisfied. Thus, Equation (9) is modified as

$$f_0(t) \cong p_0(t)\Big/\int_0^{+\infty} p_0(u)du, \tag{10}$$

and the Cumulative Distribution Function (CDF) of FPT for model M_0 is

$$F_0(t) \cong \int_0^t p_0(u)du\Big/\int_0^{+\infty} p_0(u)du, \tag{11}$$

Supposed that $\Lambda(t;\theta) = t^\theta$ and $\tau(t;\gamma) = t^\gamma$, then Equation (10) is

$$f_0(t) = \frac{(\omega\gamma - (\gamma - \theta)\mu t^\theta)}{t\sqrt{2\pi\sigma^2 t^\gamma}}\exp\left(-\frac{(\omega - \mu t^\theta)^2}{2\sigma^2 t^\gamma}\right), \tag{12}$$

When $\theta = \gamma = 1$, Equation (12) is a inverse Gaussian distribution as in Equation (4). When $\Lambda(t;\theta) = \tau(t;\gamma)$, Equation (12) is a inverse Gaussian distribution with time-scale transformation as in Equation (5). Thus, Equation (10) is the generalized PDF of FPT which can cover existing linear and time-scale transformation Wiener process model as its limiting cases.

3 PARAMETER ESTIMATION

In this section we briefly provide a two stage Maximum Likelihood Estimation (MLE) for unknown parameters in CSADT. The unknown parameters are $\Theta = \{\theta, \gamma, \sigma, \eta_0, \eta_1\}$.

Let X_{ijk} is the kth degradation value of unit j at the stress level i and t_{ijk} is the corresponding measurement time, $i = 1,2,\ldots,K$; $j = 1,2,\ldots,n_i$; $k = 1,2,\ldots,m_{ij}$. Let $\mathbf{X}_{ij} = (X_{ij1}, X_{ij2}, \ldots, X_{ijm_{ij}})'$ and $\mathbf{t}_{ij} = (\Lambda(t_{ij1};\theta), \Lambda(t_{ij2};\theta), \ldots, \Lambda(t_{ijm_{ij}};\theta))'$. According to the properties of Wiener process, X_{ij} follows a multivariate normal distribution

$$\mathbf{X}_{ij} \sim N\left(\mu_{ij}\mathbf{t}_{ij}, \sigma^2\mathbf{Q}_{ij}\right) \tag{13}$$

where

$$\mathbf{Q}_{ij} = \begin{bmatrix} \tau(t_{ij1};\gamma) & \tau(t_{ij1};\gamma) & \cdots & \tau(t_{ij1};\gamma) \\ \tau(t_{ij1};\gamma) & \tau(t_{ij2};\gamma) & \cdots & \tau(t_{ij2};\gamma) \\ \vdots & \vdots & \ddots & \vdots \\ \tau(t_{ij1};\gamma) & \tau(t_{ij2};\gamma) & \cdots & \tau(t_{ijm_{ij}};\gamma) \end{bmatrix}$$

Let $\mu = (\mu_{11}, \ldots, \mu_{1n_1}, \ldots, \mu_{Kn_K})$. The likelihood function of the CSADT data can be easily obtained and the logarithm function is

$$l(\mu,\sigma,\theta,\gamma \mid \mathbf{X}) = -\frac{\ln(2\pi)}{2}\sum_{i=1}^{K}\sum_{j=1}^{n_i} m_{ij} - \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\ln\left|\sigma^2\mathbf{Q}_{ij}\right| - \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\left(\mathbf{X}_{ij} - \mu_{ij}\mathbf{t}_{ij}\right)'\sigma^{-2}\mathbf{Q}_{ij}^{-1}\left(\mathbf{X}_{ij} - \mu_{ij}\mathbf{t}_{ij}\right) \tag{14}$$

Taking the first partial derivative of Equation (14) to μ_{ij} and σ^2 and set them equal to zero. Then, the estimation of $\hat{\mu}_{ij}$ and $\hat{\sigma}^2$ relied on θ and γ are

$$\hat{\mu}_{ij} = \frac{\mathbf{X'}_{ij}\,\mathbf{Q}_{ij}^{-1}\mathbf{t}_{ij}}{\mathbf{t'}_{ij}\,\mathbf{Q}_{ij}^{-1}\mathbf{t}_{ij}} \tag{15}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{K}\sum_{j=1}^{n_i}\left(\mathbf{X}_{ij} - \hat{\mu}_{ij}\mathbf{t}_{ij}\right)'\mathbf{Q}_{ij}^{-1}\left(\mathbf{X}_{ij} - \hat{\mu}_{ij}\mathbf{t}_{ij}\right)}{\sum_{i=1}^{K}\sum_{j=1}^{n_i} m_{ij}} \tag{16}$$

Substituting Equation (15) and (16) to (14), the log-likelihood function is only a function of θ and γ, i.e.

$$l(\theta,\gamma \mid \mathbf{X}) = -\frac{\ln(2\pi)}{2}\sum_{i=1}^{K}\sum_{j=1}^{n_i} m_{ij} - \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\ln\left|\hat{\sigma}^2\mathbf{Q}_{ij}\right| - \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{n_i}(\mathbf{X}_{ij} - \hat{\mu}_{ij}\mathbf{t}_{ij})'\hat{\sigma}^{-2}\mathbf{Q}_{ij}^{-1}(\mathbf{X}_{ij} - \hat{\mu}_{ij}\mathbf{t}_{ij}) \tag{17}$$

Thus, $\hat{\theta}$ and $\hat{\gamma}$ can be obtained by the two-dimensional search for the maximum value of Equation (17) (Lagarias et al. 1998). Then, the estimation of $\hat{\mu}_{ij}$ and $\hat{\sigma}^2$ can be easily computed by substituting $\hat{\theta}$ and $\hat{\gamma}$ to Equation (15) and (16).

Given the $\hat{\mu}_{ij}$s and the corresponding accelerated stress I, the values of $\hat{\eta}_0$ and $\hat{\eta}_1$ can be easily computed with Equation (3) by the MLE method.

4 CASE STUDY

In this section, the LED case is used to illustrate the proposed method in nonlinear ADT analysis, see Figure 1. In order to compare the model M_0 with the time-scale transformation model M_1 as used in Tang et al. (2014), the Akaike information criterion (AIC) is selected

$$AIC = -2l_{max} + 2p, \tag{18}$$

where l_{max} is the maximum value of the log-likelihood function in Equation (14); p is the number of unknown parameters, $p = 5$ in the LED case.

For convenience, the formulas that $\Lambda(t;\theta) = t^{\theta}$ and $\tau(t;\gamma) = t^{\gamma}$ are used in this paper. The estimates of unknown parameters for model M_0 and M_1 are given in Table 1.

As seen from Table 1, model M_0 fits better than model M_1 with a larger l_{max} and lower AIC values. Figure 2 also indicates that M_0 performs better than M_1 when fitting the LED data at two stress levels. For the estimates of parameters in degradation model, θ is slightly different with 0.4415 in M_0 and 0.4503 in M_1, which means that both of the two models claim the similar degradation path for the LED case because the expectation is $E[X(t)] = \mu t^{\theta}$.

On the contrary, the values of γ and σ^2 is quite different from each other. In model M_1, the parameter γ is set to be equal with θ. Hence, its σ^2 is smaller than that in model M_0 since that its γ value is larger than model M_0, i.e. $0.4503 > 0.1172$, which means this two models provide different results of the variance of the degradation process in LED case ($Var(X(t)) = \sigma^2 t^{\gamma}$). In general, it may prefer to choose the case with lower σ^2 and its PDF of FPT will be shaper. However, in this application, model M_0 is applicable since it can better describe the ADT data. It is to say that model M_1 underestimates the noise level of the degradation process, which then effects the estimates of the parameters in acceleration model. Both η_0 and η_1 are significantly different in two models. Furthermore, the lifetime evaluation results will be quite different.

As discussed before, the 95% confidence interval of the MTTF at the normal stress under 25 mA is [1848, 57202] hours for model M_1 (Tang et al. 2014), while the result from the original paper is about 1346 hours (Chaluvadi 2008). We call this situation "the trap of extrapolation".

The PDFs and CDFs of FPT for model M_0 and M_1 at normal stress are shown in Figures 3 and 4. The 95% confidence intervals of MTTF are [232, 2622] and [1914, 53629] hours, while the mean values are 1100 and 12941 hours, respectively. Hence, model M_0 with its generalized formula can provide more reliable lifetime evaluation results than M_1 after the extrapolation from higher stress conditions to normal condition.

†The trap of extrapolation

In order to analyze the trap of extrapolation, we compare the estimated drift coefficients for two models since they are related to the stress conditions and used for extrapolation. The results are shown in Figure 5. The median values of the drift coefficients at two accelerated stress conditions are 3.0768/3.4834 and 2.3090/3.4946 for model M_0 and M_1, respectively. In addition, the estimated acceleration models are $\mu = 0.2284 I^{0.7257}$ and $\mu = 1.8038 \times 10^{-5} I^{3.2968}$. Thus, the extrapolated drift coefficients at 25 mA, i.e. μ_0, are 2.3616 and 0.7327 accordingly.

Obviously, the time-scale transformation model M_1 has a significantly lower value of μ_0 than model M_0, thus to compute a quite optimistic lifetime evaluation result which will give false confidence to the producers on this products. If such products are released to the market, the following unexpected maintenance may lead to severe consequence. For the LED case, the improper linearlization of the ADT data by model M_1 cannot be used for further lifetime and reliability evaluation. It is interesting to unearth the true meaning of ADT analysis since that different modeling methods may lead to significantly different results. In the work of Meeker and Escobar (1998), this problem is classified into the PITFALL 3 of the accelerated

Table 1. Unknown parameters for M_0 and M_1 in LED case.

	Degradation model			Acceleration model			
Models	θ	γ	σ^2	η_0	η_1	l_{max}	AIC
M_0	0.4415	0.1172	73.7836	0.2284	0.7257	–310.3884	630.7768
M_1	0.4503	0.4503	5.8400	1.8038e-05	3.2968	–316.9837	641.9673

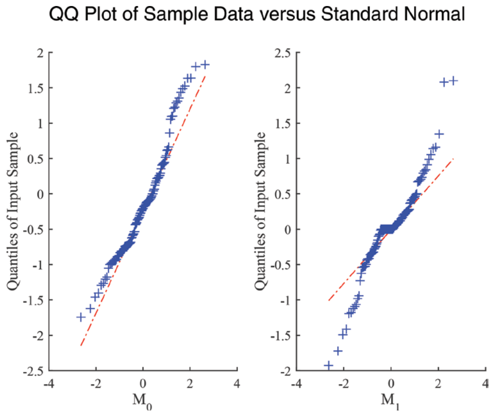

Figure 2. Goodness of fit for two models in the LED case.

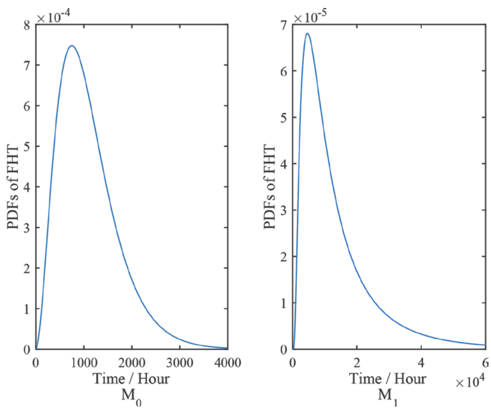

Figure 3. The PDFs of FPT for two models in the LED case.

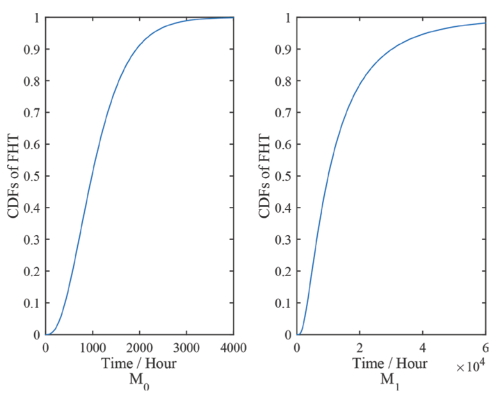

Figure 4. The CDFs of FPT for two models in the LED case.

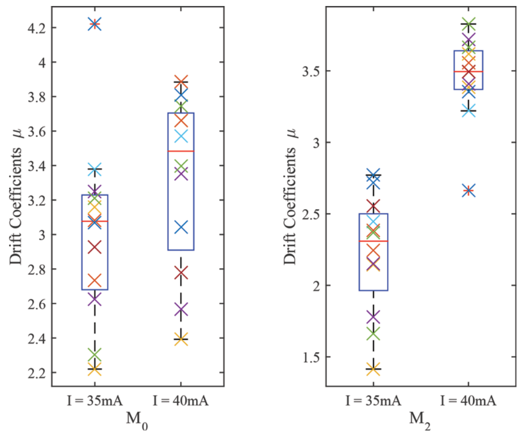

Figure 5. The drift coefficients for two models in the LED case.

testing: "Multiple time-scales and multiple factors affecting degradation". Hence, the assessments results should be accepted with caution after comprehensively analyzing the nonlinear ADT data, especially to the extrapolation from high stress conditions to normal condition.

One feasible way to avoid this problem may be to increase the number of stress conditions in ADT. With data in multiple stress conditions rather than two in the LED case, the estimation error of the parameters in acceleration model will be decreased, the same to the estimation of μ_0.

5 CONCLUSIONS

In this work, a general Wiener process is introduced to analyze the nonlinear accelerated degradation data. The proposed method can cover the common used linear and time-scale transformation Wiener processes as its limiting cases. The LED case shows that this method fits better than time-scale transformation model and can provide reliable lifetime estimation results. Meanwhile, it can avoid the tap of extrapolation.

However, more applications are needed to verify the effectiveness of this method on nonlinear ADT analysis. Further research may be given to: the unit-to-unit variation (see Fig. 5), the multiple accelerated variables, the nonlinear SSADT data and the optimum plan design.

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